Journal of Global Optimization (2005) 32: 467–484
DOI 10.1007/s10898-003-2684-1

Semicontinuous Mappings into T.V.S. with Applications to Mixed Vector Variational-Like Inequalities★

This paper is dedicated to Professor Franco Giannessi for his 68th birthday

Y. CHIANG
Department of Applied Mathematics, National Sun Yat-sen University, Kaohsiung 80424, Taiwan, R.O.C.

(Received 15 August 2003; accepted 19 August 2003)

Abstract. Let $\mathcal{L}(X,\mathcal{Z})$ be the space of continuous linear mappings between topological vector spaces, where $\mathcal{Z}$ is Hausdorff and preordered by a closed convex cone C. In this paper, we introduce a notion of semicontinuity to any function from a topological space into X. A notion of semicontinuity is also introduced to any function from a topological space into $\mathcal{L}(X,\mathcal{Z})$. These two notions of semicontinuity are related by the embedding of X into $\mathcal{L}(X,\mathcal{Z})$. Their basic properties are given. As an application, we derive some existence results for the mixed vector variational-like inequality.

Key words: Closed convex bounded base of a cone, C-semicontinuous functions, mixed vector variational-like inequalities, the topologies of simple convergence and bounded convergence.

1. Introduction

By a topological vector space we shall always mean a real topological vector space. For any two topological vector spaces X and $\mathcal{Z}$, let $\mathcal{L}(X,\mathcal{Z})$ denote the family of all continuous linear mappings from X into $\mathcal{Z}$. When $\mathcal{Z}$ is the set $\mathbb{R}$ of all real numbers, $\mathcal{L}(X,\mathcal{Z})$ is the usual topological dual space X^* of X. For any $x \in X$ and any $\ell \in \mathcal{L}(X,\mathcal{Z})$, we shall alternatively write the value $\ell(x)$ as $\langle \ell, x\rangle$.

Throughout the paper, let $\mathcal{Z}$ denote an ordered Hausdorff topological vector space with a preordering defined by a closed convex cone $C \subset \mathcal{Z}$ such that $C \neq \mathcal{Z}$ and $\operatorname{int} C \neq \emptyset$, where $\operatorname{int} C$ is the interior of C in $\mathcal{Z}$. Note that $C \neq \mathcal{Z}$ if and only if $\operatorname{int} C$ does not contain the zero vector.

The work was motivated by solving the following mixed vector variational-like inequality. For a nonempty subset K of a topological vector

★This work was partially supported by grants from the National Science Council of the Republic of China.

space X, the mixed vector variational-like inequality problem associated with the given functions $T: K \longrightarrow \mathcal{L}(X, \mathcal{Z}), \varphi: K \times K \longrightarrow X$ and $f: K \times K \longrightarrow \mathcal{Z}$, MVVLI$(T, \varphi, f)$ for short, is the problem to find $\hat{x} \in K$ such that

$$\langle T\hat{x}, \varphi(\hat{x}, y)\rangle + f(\hat{x}, y) \in (-\operatorname{int} C)^c \quad \text{for all } y \in K, \tag{1}$$

where $(-\operatorname{int} C)^c$ is the complement of $-\operatorname{int} C$ in $\mathcal{Z}$.

The modeling of the problem (1) is quite general. When $f \equiv 0$ and $\varphi(x, y) = y - x$, the problem (1) becomes the vector variational inequality. The vector variational inequality was first introduced by Giannessi in finite dimensional spaces (Giannessi, 1980), and later was generalized and extensively studied. See Chen and Yang (1990), Chiang and Yao (2002), Konnov (2001), Yang and Yao (2002) and references there in.

When $\mathcal{Z} = \mathbb{R}$ and $C = \mathbb{R}_+$ the set of all non-negative real numbers, the MVVLI(T, φ, f) becomes the mixed variational-like inequality problem of finding $\hat{x} \in K$ such that

$$\langle T\hat{x}, \varphi(\hat{x}, y)\rangle + f(\hat{x}, y) \geqslant 0 \quad \text{for all } y \in K. \tag{2}$$

When $f \equiv 0$ and $\varphi(x, y) = y - x$, the problem (2) is the usual variational inequality. If $f(x, y) = g(y) - g(x)$ for some function $g: K \longrightarrow \mathbb{R}$, then the problem (2) reduces to that studied in Ansari and Yao (2001), Dien (1992) and Noor (1994). If $f \equiv 0$, then the problem (2) becomes the variational-like inequality studied in Ansari and Yao (1998), Ansari and Yao (2000), Parida et al. (1989), Siddiqi et al. (1994) and Yang and Chen (1992).

It is well known that there is a very close connection between optimization problems and variational inequalities. It turns out that the vector variational inequality also provides a very good and useful tool in dealing with vector optimization problems. See Chen and Yang (1990), Chen (1989), Chen and Cheng (1987), Lee et al. (2000), Yang and Goh (1997) and the references therein. It is then worth efforts to pay much more attention to the research on vector variational inequalities.

Up to the author's knowledge, in most work on variational-like inequalities, the *weakest* continuity assumption which would be made on a function $g: K \longrightarrow X$ was to require that g is continuous from the strong topology on K to the weak topology on X. This equivalent to requiring that $\ell \circ g$ is continuous from K into $\mathbb{R}$ for every $\ell \in X^*$. Here, we shall consider the case where $\ell \circ g$ is upper (or lower) semicontinuous for very $\ell \in X^*$.

More generally, in Section 3, we shall consider any function g from a topological space Y into X such that for every $\ell \in \mathcal{L}(X, \mathcal{Z})$ the function $\ell \circ g$ is C-upper semicontinuous or C-lower semicontinuous in the sense of Tanaka (Tanaka, 1997). Such a function g will be called $C_{\mathcal{L}}$-upper or $C_{\mathcal{L}}$-lower semicontinuous.

By considering the $\mathcal{L}$-topology introduced in Chiang and Yao (2002), we prove in Theorem 3.1 that if C has a closed convex bounded base (see Remark 3.1), then $g: Y \longrightarrow X_{\mathcal{L}}$ is continuous if and only if g is simultaneously $C_{\mathcal{L}}$-upper semicontinuous and $C_{\mathcal{L}}$-lower semicontinuous, where $X_{\mathcal{L}}$ is the space X equipped with the $\mathcal{L}$-topology. The $\mathcal{L}$-topology will be recalled in Section 2.

To define the continuity of the function T given in (1), we consider two topologies on $\mathcal{L}(X, \mathcal{Z})$, the topology of simple convergence and the topology of bounded convergence. These topologies are described in Section 2. We shall write $\mathcal{L}_{\mathcal{E}}(X, \mathcal{Z})$ for the space $\mathcal{L}(X, \mathcal{Z})$ either equipped with the topology of simple convergence or of bounded convergence.

In Section 3, we introduce a notion of $C_{\mathcal{E}}^*$-semicontinuity to any function T from a topological space W into $\mathcal{L}_{\mathcal{E}}(X, \mathcal{Z})$ so that T is continuous if and only if T is simultaneously $C_{\mathcal{E}}^*$-upper semicontinuous and $C_{\mathcal{E}}^*$-lower semicontinuous whenever C has a closed convex bounded base; see Theorem 3.4. By using the embedding from X into $\mathcal{L}(X, \mathcal{Z})$, the notion of $C_{\mathcal{E}}^*$-semicontinuity is related to the notion of $C_{\mathcal{L}}$-semicontinuity in Theorem 3.6.

To find solutions for the mixed vector variational-like inequality problem (1), we prove that if $T: W \longrightarrow \mathcal{L}_{\mathcal{E}}(X, \mathcal{Z})$ is $C_{\mathcal{E}}^*$-upper semicontinuous and if $g: Y \longrightarrow X$ is $C_{\mathcal{L}}$-upper semicontinuous, then, under certain conditions, the function $(w, y) \longmapsto \langle T(w), g(y) \rangle$ is C-upper semicontinuous on $W \times Y$. See Theorem 3.7 and its corollaries. With these theorems, we derive some existence results in Section 4 for the problem (1) with the associated functions semicontinuous.

In this paper, unless specifically stated otherwise, we shall always refer to X as a topological vector space. For a nonempty subset A of X, let $\text{co}(A)$ denote the convex hull of A, and let $\mathcal{F}(A)$ denote the family of all nonempty finite subsets of A.

2. Preliminaries

In this section, we recall the $\mathcal{L}$-topology defined in Chiang and Yao (2002), the topology of simple convergence and the topology of bounded convergence on $\mathcal{L}(X, \mathcal{Z})$.

The $\mathcal{L}$-topology on X is the topology having the sets $\ell^{-1}(U)$ as subbasis elements, where U is open in $\mathcal{Z}$ and $\ell \in \mathcal{L}(X, \mathcal{Z})$. When $\mathcal{Z} = \mathbb{R}$, the $\mathcal{L}$-topology on X becomes the usual weak topology. Let $X_{\mathcal{L}}$ denote the space X equipped with the $\mathcal{L}$-topology. It is easy to see that $X_{\mathcal{L}}$ is a topological vector space. Note that $X_{\mathcal{L}}$ is Hausdorff if X is Hausdorff and locally convex (Chiang and Yao, 2002, Theorem 3.1).

A subset E of X will be called $\mathcal{L}$-open ($\mathcal{L}$-closed, $\mathcal{L}$-bounded or $\mathcal{L}$-compact) if E is open (closed, bounded or compact) in $X_{\mathcal{L}}$. It is clear that every $\mathcal{L}$-open (or $\mathcal{L}$-closed) subset of X is originally open (or closed) in X, but not conversely in general. Similarly, compact (or bounded) subsets of X are $\mathcal{L}$-compact (or $\mathcal{L}$-bounded). We shall prove in Proposition 2.2 that if X is a normed space and $\mathcal{Z}$ is a Banach space, then $\mathcal{L}$-bounded subsets are bounded.

A net $\{x_\alpha\}$ in X is called $\mathcal{L}$-convergent if it is convergent in $X_{\mathcal{L}}$ to some $x \in X$, denoted by $x_\alpha \overset{\mathcal{L}}{\to} x$. Observe that $x_\alpha \overset{\mathcal{L}}{\to} x$ if and only if $\langle \ell, x_\alpha \rangle \to \langle \ell, x \rangle$ for all $\ell \in \mathcal{L}(X, \mathcal{Z})$. When $\mathcal{Z} = \mathbb{R}$, the $\mathcal{L}$-convergence coincides with the weak convergence.

PROPOSITION 2.1. *A nonempty set $K \subset X$ is $\mathcal{L}$-bounded if and only if $\ell(K)$ is bounded in $\mathcal{Z}$ for every $\ell \in \mathcal{L}(X, \mathcal{Z})$.*

Proof. Clearly, $\ell(K)$ is bounded in $\mathcal{Z}$ when K is $\mathcal{L}$-bounded. Conversely, assume that $\ell(K)$ is bounded for every $\ell \in \mathcal{L}(X, \mathcal{Z})$, and consider any 0-neighborhood U in $X_{\mathcal{L}}$. Write

$$U = \bigcap_{j=1}^{m} \ell_j^{-1}(V_j),$$

where $\ell_j \in \mathcal{L}(X, \mathcal{Z})$ and V_j are 0-neighborhoods in $\mathcal{Z}$ for $1 \leqslant j \leqslant m$. Choose a balanced 0-neighborhood V in $\mathcal{Z}$ such that

$$V \subset \bigcap_{j=1}^{m} V_j.$$

For every j, there is a $\lambda_j > 0$ such that $\ell_j(K) \subset \lambda_j V$. By setting $\lambda = \max_{1 \leqslant j \leqslant m} \lambda_j$, we have $\ell_j(K) \subset \lambda_j V \subset \lambda V$, and

$$K \subset \lambda \bigcap_{j=1}^{m} \ell_j^{-1}(V) \subset \lambda \bigcap_{j=1}^{m} \ell_j^{-1}(V_j) = \lambda U.$$

This proves that K is $\mathcal{L}$-bounded. □

Let $\mathcal{B}_X$ denote the family of all bounded subsets of X, and let $\mathcal{N}_{\mathcal{Z}}$ be the family of 0-neighborhoods in $\mathcal{Z}$. For every $E \in \mathcal{B}_X$ and every $V \in \mathcal{N}_{\mathcal{Z}}$, let

$$[E, V] = \{f \in \mathcal{L}(X, \mathcal{Z}) : f(E) \subset V\}.$$

Let $\mathcal{F}_0(X) = \mathcal{F}(X) \cup \{\emptyset\}$. For $\mathcal{E} = \mathcal{B}_X$ or $\mathcal{F}_0(X)$, there is a unique translation-invariant topology $\mathcal{T}_{\mathcal{E}}$ on $\mathcal{L}(X, \mathcal{Z})$ so that the topological vector space $\mathcal{L}_{\mathcal{E}}(X, \mathcal{Z}) = (\mathcal{L}(X, \mathcal{Z}), \mathcal{T}_{\mathcal{E}})$ has the family

$$\{[E, V]: E \in \mathcal{E} \text{ and } V \in \mathcal{N}_{\mathcal{Z}}\}$$

as its 0-neighborhood base. See Schaefer (1999), p. 79.

(a) If $\mathcal{E} = \mathcal{F}_0(X)$, then $\mathcal{T}_{\mathcal{E}}$ is the topology of simple convergence (or the topology of pointwise convergence). In this case, we write $\mathcal{L}_{\mathcal{E}}(X, \mathcal{Z}) = \mathcal{L}_s(X, \mathcal{Z})$. When $\mathcal{Z} = \mathbb{R}, \mathcal{L}_s(X, \mathcal{Z})$ is the usual weak-star topology on X^*.
(b) If $\mathcal{E} = \mathcal{B}_X$, then $\mathcal{T}_{\mathcal{E}}$ is the topology of bounded convergence. In this case, we write $\mathcal{L}_{\mathcal{E}}(X, \mathcal{Z}) = \mathcal{L}_b(X, \mathcal{Z})$.

Note that $\mathcal{L}_s(X, \mathcal{Z})$ and $\mathcal{L}_b(X, \mathcal{Z})$ are Hausdorff since $\mathcal{Z}$ is Hausdorff (Schaefer, 1999, pp. 79–80). If X and $\mathcal{Z}$ are normed spaces, the norm

$$\ell \longmapsto \|\ell\| = \sup\{|\langle \ell, x\rangle| : |x| \leqslant 1\}$$

generates that topology of bounded convergence on $\mathcal{L}(X, \mathcal{Z})$, i.e., $\mathcal{L}_b(X, \mathcal{Z})$ is also a normed space (Schaefer, 1999, p. 81). Moreover, $\mathcal{L}_b(X, \mathcal{Z})$ is a Banach space if $\mathcal{Z}$ is a Banach space (Schaefer, 1999, p. 42).

It is well known that every $x \in X$ induces a continuous linear mapping x^* from $\mathcal{L}_b(X, \mathcal{Z})$ into $\mathcal{Z}$ defined by

$$\langle x^*, y\rangle = \langle y, x\rangle \quad \text{for } y \in \mathcal{L}(X, \mathcal{Z}).$$

Note that the mapping $J: X \longrightarrow \mathcal{L}_b(\mathcal{L}_b(X, \mathcal{Z}), \mathcal{Z})$ defined by

$$J(x) = x^*$$

is an embedding, and that if X and $\mathcal{Z}$ are normed spaces, then $\|J(x)\| = |x|$ for $x \in X$. As a consequence of the principle of uniform boundedness (Schaefer, 1999, p. 84), we prove:

PROPOSITION 2.2. *Let X be a normed space and $\mathcal{Z}$ be a Banach space. Then a nonempty subset K of X is $\mathcal{L}$-bounded if and only if it is bounded.*

Proof. It suffices to show that K is bounded when it is $\mathcal{L}$-bounded. Let $K^* = \{x^* : x \in K\}$. It follows from Proposition 2.1 that for every $\ell \in \mathcal{L}_b(X, \mathcal{Z})$,

$$K^*(\ell) = \{\langle x^*, \ell\rangle : x \in K\} = \{\langle \ell, x\rangle : x \in K\} = \ell(K)$$

is bounded in $\mathcal{Z}$. Since $\mathcal{L}_b(X, \mathcal{Z})$ is a Banach space, from the principle of uniform boundedness, we conclude that K^* is bounded in $\mathcal{L}_b(\mathcal{L}_b(X, \mathcal{Z}), \mathcal{Z})$, and so K is bounded in X. □

Let $\{y_\alpha\}$ be a net in $\mathcal{L}(X,\mathcal{Z})$. We shall simply write $y_\alpha \longrightarrow y$ when $\{y_\alpha\}$ converges in $\mathcal{L}_b(X,\mathcal{Z})$ to $y \in \mathcal{L}(X,\mathcal{Z})$, and write $y_\alpha \stackrel{s}{\longrightarrow} y$ when $\{y_\alpha\}$ converges in $\mathcal{L}_s(X,\mathcal{Z})$ to y. Note that $y_\alpha \stackrel{s}{\longrightarrow} y$ if and only if $\langle y_\alpha, x\rangle \to \langle y, x\rangle$ for every $x \in X$. When $\mathcal{Z} = \mathbb{R}$, the simple convergence is the usual weak-star convergence.

The proof of the following proposition is routine and omitted.

PROPOSITION 2.3. *Assume that X is Hausdorff. Let* $\{(x_\alpha, \ell_\alpha)\}$ *be a net in* $X \times \mathcal{L}(X,\mathcal{Z})$. *Then* $\lim_\alpha \langle \ell_\alpha, x_\alpha\rangle = \langle \ell, x\rangle$ *if either one of the following holds.*

(i) $\{x_\alpha\}$ *lies in a bounded subset of X with* $x_\alpha \stackrel{\mathcal{L}}{\longrightarrow} x \in X$, *and* $\ell_\alpha \longrightarrow \ell \in \mathcal{L}(X,\mathcal{Z})$.

(ii) $x_\alpha \longrightarrow x$, *and* $\{\ell_\alpha\}$ *lies in a bounded subset of* $\mathcal{L}_b(X,\mathcal{Z})$ *with* $\ell_\alpha \stackrel{s}{\longrightarrow} \ell \in \mathcal{L}(X,\mathcal{Z})$.

REMARK 2.1. As a consequence of Corollary 4.1 and Theorem 4.2 (p. 83) in Schaefer (1999), if either X and $\mathcal{Z}$ are Hausdorff locally convex with X barreled or X is a Baire space, then, in (ii) of Proposition 2.3, it is enough to require that $\{\ell_\alpha\}$ is bounded in $\mathcal{L}_s(X,\mathcal{Z})$.

3. Semicontinuous Mappings

In this section, we shall introduce semicontinuity to functions from topological spaces into X or $\mathcal{L}(X,\mathcal{Z})$. Recall that a function f form a topological space Y into $\mathcal{Z}$ is C-upper semicontinuous (Tanaka, 1997) if $f^{-1}(z - \operatorname{int} C)$ is open in Y for every $z \in \mathcal{Z}$, or equivalently, for every $y \in Y$ and for every $v \in \operatorname{int} C$ there is a neighborhood U of y such that

$$f(y') \in f(y) + v - \operatorname{int} C \text{ for all } y' \in U.$$

While f is C-lower semicontinuous if $-f$ is C-upper semicontinuous.

When $(\mathcal{Z}, C) = (\mathbb{R}, \mathbb{R}_+)$, the notion of C-upper (or C-lower) semicontinuity reduces to the usual upper (or lower) semicontinuity. It is well known that a function $f: Y \to \mathbb{R}$ is upper semicontinuous if and only if lim $\sup_\alpha f(y_\alpha) \leqslant f(y)$ whenever $\{y_\alpha\}$ is a net in Y converging to $y \in Y$. There is an analogous criterion for C-upper semicontinuity proved in Theorem 2.4 of Chadli et al. (2002), which says that, when Y is Hausdorff, a function $f: Y \to \mathcal{Z}$ is C-upper semicontinuous if and only if for every $v \in \operatorname{int} C$ and for every $y \in Y$, there is an α_v (depending on x) such that

$$\alpha \succeq \alpha_v \Rightarrow \overline{\{f(y_\beta) : \beta \succeq \alpha\}} \subset f(y) + v - \operatorname{int} C,$$

whenever $\{y_\alpha\}$ is a net in Y converging to y.

REMARK 3.1. A mapping from a topological space into $\mathcal{Z}$ may not be continuous when it is simultaneous C-upper and C-lower semicontinuous. See Remark 5.4 (p. 23) in Luc (1989) for an example. Such a mapping is continuous when C has a closed convex bounded base (Luc, 1989, Theorem 5.3, p. 22). A base B of C is a subset of $\mathcal{Z}-\{0\}$ such that $C=\{tb: b\in B$ and $t\in \mathbb{R}_+\}$, and such that for every $c\in C-\{0\}$, there is a unique $(b,t)\in B\times\mathbb{R}_+$ with $c=tb$. When $\mathcal{Z}$ is finite dimensional, C has a closed convex bounded base if and only if it is pointed, i.e., $C\cap(-C)=\{0\}$. However, this is in general not true in infinite dimensional spaces; see Remark 5.4 (p. 23) in Luc (1989).

Note that if a closed cone C in a Hausdorff topological vector space $\mathcal{Z}$ has a closed convex bounded base, then C is pointed (Luc, 1989, Proposition 1.7, p. 4), and for any given 0-neighborhood V in $\mathcal{Z}$ there is another 0-neighborhood V_0 in $\mathcal{Z}$ such that $(V_0-C)\cap(V_0+C)\subset V$ (Luc, 1989, Proposition 1.8, p. 5).

DEFINITION 3.1. Let Y be a topological space. A function $f: Y\to X$ will be called *$C_{\mathcal{L}}$-upper semicontinuous* if for every $\ell\in\mathcal{L}(X,\mathcal{Z})$, the function $\ell\circ f$ is C-upper semicontinuous. If $-f$ is $C_{\mathcal{L}}$-upper semicontinuous, then f is called *$C_{\mathcal{L}}$-lower semicontinuous*.

It is clear that if f is continuous from Y into X or $X_{\mathcal{L}}$, then f is simultaneously $C_{\mathcal{L}}$-upper and $C_{\mathcal{L}}$-lower semicontinuous.

THEOREM 3.1. *Let f be a function from a topological space Y into X, and assume that C has a closed convex bounded base. Then $f: Y\to X_{\mathcal{L}}$ is continuous if and only if f is simultaneously $C_{\mathcal{L}}$-upper semicontinuous and $C_{\mathcal{L}}$-lower semicontinuous.*

Proof. Assume that f is $C_{\mathcal{L}}$-upper semicontinuous and is $C_{\mathcal{L}}$-lower semicontinuous. Note that f is continuous from Y into $X_{\mathcal{L}}$ if and only if $\ell\circ f$ is continuous from Y into $\mathcal{Z}$ for every $\ell\in\mathcal{L}(X,\mathcal{Z})$. Since $\ell\circ f$ is simultaneously C-upper semicontinuous and C-lower semicontinuous, and since C has a closed convex bounded base, $\ell\circ f$ is continuous. □

The following theorem is a consequence of Theorem 2.1 of Tanaka (1997).

THEOREM 3.2. *Let Y be a topological space, and let $f,g: Y\to X$ and $\mu: Y\to\mathbb{R}$ be functions such that $\mu(y)\geqslant 0$ for all $y\in Y$.*

(i) *if f and g are $C_{\mathcal{L}}$-upper semicontinuous, then so are $f+g$, and λf for any $\lambda\geqslant 0$.*
(ii) *If f is $C_{\mathcal{L}}$-upper semicontinuous, and if μ is upper semicontinuous, then for any fixed point x_0 of X the function $h: Y\to X$ defined by $h(y)=f(y)+\mu(y)x_0$ is $C_{\mathcal{L}}$-upper semicontinuous.*

Proof. Let $\ell \in \mathcal{L}(X, \mathcal{Z})$ be arbitrary. Since $\ell \circ f$ and $\ell \circ g$ are C-upper semicontinuous, $\ell \circ (f+g) = \ell \circ f + \ell \circ g$ and $\ell \circ (\lambda f) = \lambda(\ell \circ f)$ are also C-upper semicontinuous.

Note that $\ell \circ h(y) = \ell \circ f(y) + \mu(y)\ell(x_0)$ for all $y \in Y$, and that the function $y \longmapsto \mu(y)\ell(x_0)$ is C-upper semicontinuous. The proof is complete. □

In the rest of this section, we shall also denote $\mathcal{E}$ by $\mathcal{B}_X$ or $\mathcal{F}_0(X) = \mathcal{F}(X) \cup \{\emptyset\}$. Now, we are going to define semicontinuity for mappings from topological spaces into $\mathcal{L}_{\mathcal{E}}(X, \mathcal{Z})$. We remark that for any $(E, v) \in \mathcal{E} \times \operatorname{int} C$, the set $[E, v - \operatorname{int} C]$ is a 0-neighborhood in $\mathcal{L}_{\mathcal{E}}(X, \mathcal{Z})$.

DEFINITION 3.2. Let Y be a topological space, and let $T: Y \to \mathcal{L}(X, \mathcal{Z})$ be a function.

(i) T will be called $C^*_{\mathcal{E}}$*-upper semicontinuous* at $y_0 \in Y$ if for any $(E, v) \in \mathcal{E} \times \operatorname{int} C$, there is a neighborhood U of y_0 such that $T(y) \in T(y_0) + [E, v - \operatorname{int} C]$ for all $y \in U$.
(ii) If $-T$ is $C^*_{\mathcal{E}}$-upper semicontinuous at y_0, then T is called $C^*_{\mathcal{E}}$*-lower semicontinuous* at y_0. Equivalently, for any $(E, v) \in \mathcal{E} \times \operatorname{int} C$, there is a neighborhood U of y_0 such that $T(y) \in T(y_0) + [E, -v + \operatorname{int} C]$ for all $y \in U$.
(iii) T is simply called $C^*_{\mathcal{E}}$-upper(respectively, $C^*_{\mathcal{E}}$-lower) semicontinuous if it is $C^*_{\mathcal{E}}$-upper (respectively, $C^*_{\mathcal{E}}$-lower) semicontinuous at every point of Y.

We shall write $C^*_{\mathcal{E}} = C^*_{\mathcal{L}(b)}$ when $\mathcal{E} = \mathcal{B}_X$, and write $C^*_{\mathcal{E}} = C^*_{\mathcal{L}(s)}$ when $\mathcal{E} = \mathcal{F}_0(X)$. Note that

(a) T is $C^*_{\mathcal{L}(s)}$-upper semicontinuous at y_0 if and only if for any $(x, v) \in X \times \operatorname{int} C$ there is a neighborhood U of y_0 such that $T(y) \in T(y_0) + [\{x\}, v - \operatorname{int} C]$ for all $y \in U$, and
(b) T is $C^*_{\mathcal{L}(s)}$-lower semicontinuous at y_0 if and only if for any $(x, v) \in X \times \operatorname{int} C$ there is a neighborhood U of y_0 such that $T(y) \in T(y_0) + [\{x\}, -v + \operatorname{int} C]$ for all $y \in U$.

REMARK 3.2. Let T be a function from a topological space Y into $\mathcal{L}(X, \mathcal{Z})$. The following assertions are immediate consequences of definition.

(a) If T is $C^*_{\mathcal{L}(b)}$-upper (respectively, $C^*_{\mathcal{L}(b)}$-lower) semicontinuous, then T is $C^*_{\mathcal{L}(s)}$-upper (respectively, $C^*_{\mathcal{L}(s)}$-lower) semicontinuous.
(b) If $T: Y \longrightarrow \mathcal{L}_{\mathcal{E}}(X, \mathcal{Z})$ is continuous, then it is $C^*_{\mathcal{E}}$-upper semicontinuous and is $C^*_{\mathcal{E}}$-lower semicontinuous.

By using a similar argument as in the proof of Proposition 2.1 of Tanaka (1997), we prove

THEOREM 3.3. *Let T be a function from a topological space Y into $\mathcal{L}(X,\mathcal{Z})$. Then:*

(i) *T is $C^*_{\mathcal{E}}$-upper semicontinuous at $y_0\in Y$ if and only if for any 0-neighborhood $[E,V]$ in $\mathcal{L}_{\mathcal{E}}(X,\mathcal{Z})$ there is a neighborhood U of y_0 such that*

$$T(y)-T(y_0)\in[E,V-\operatorname{int}C]\ \textit{for all}\ y\in U.$$

(ii) *T is $C^*_{\mathcal{E}}$-lower semicontinuous at $y_0\in Y$ if and only if for any 0-neighborhood $[E,V]$ in $\mathcal{L}_{\mathcal{E}}(X,\mathcal{Z})$ there is a neighborhood U of y_0 such that*

$$T(y)-T(y_0)\in[E,V+\operatorname{int}C]\ \textit{for all}\ y\in U.$$

Proof. Note that any 0-neighborhood V in $\mathcal{Z}$ contains a symmetric 0-neighborhood V', i.e., $V'=-V'$. Since

$$[E,-V'+\operatorname{int}C]\subset[E,V+\operatorname{int}C],$$

by considering symmetric 0-neighborhoods in $\mathcal{Z}$, the statement (ii) follows from (i) immediately.

For the proof of (i), we first assume that T is $C^*_{\mathcal{E}}$-upper semicontinuous. Choose any $v\in V\cap\operatorname{int}C$. There is a neighborhood U of y_0 such that

$$y\in U\Longrightarrow T(y)-T(y_0)\in[E,v-\operatorname{int}C]\subset[E,V-\operatorname{int}C].$$

Conversely, let $v\in\operatorname{int}C$ be arbitrary. There is a neighborhood V_0 of v in $\mathcal{Z}$ such that $V_0\subset\operatorname{int}C$. Clearly, $V=v-V_0$ is a 0-neighborhood in $\mathcal{Z}$. Note that

$$V-\operatorname{int}C=v-V_0-\operatorname{int}C\subset v-\operatorname{int}C$$

since $V_0\subset\operatorname{int}C$. There is a neighborhood U of y_0 such that

$$y\in U\Longrightarrow T(y)-T(y_0)\in[E,V-\operatorname{int}C]\subset[E,v-\operatorname{int}C].$$

Therefore, T is $C^*_{\mathcal{E}}$-upper semicontinuous at y_0. □

THEOREM 3.4. *Let T be a function from a topological space Y into $\mathcal{L}(X,\mathcal{Z})$, and assume that C has a closed convex bounded base. Then $T\colon Y\to\mathcal{L}_{\mathcal{E}}(X,\mathcal{Z})$ is continuous if and only if it is simultaneously $C^*_{\mathcal{E}}$-upper semicontinuous and $C^*_{\mathcal{E}}$-lower semicontinuous.*

Proof. It remains to show that T is continuous when it is simultaneously $C^*_{\mathcal{E}}$-upper semicontinuous and $C^*_{\mathcal{E}}$-lower semicontinuous.

Let $y_0 \in Y$ be arbitrary, and let $[E, V]$ be any 0-neighborhood in $\mathcal{L}_{\mathcal{E}}(X, \mathcal{Z})$. It follows from Proposition 1.8 (p. 5) in Luc (1989) that there is a 0-neighborhood V_0 in $\mathcal{Z}$ such that

$$(V_0 - C) \cap (V_0 + C) \subset V;$$

(see Remark 3.1). By Theorem 3.3, there is neighborhood U of y_0 such that

$$y \in U \Longrightarrow T(y) - T(y_0) \in [E, V_0 - \operatorname{int} C] \cap [E, V_0 + \operatorname{int} C] \subset [E, V].$$

The proof is complete. □

THEOREM 3.5. *Let S and T be operators from a topological space Y into $\mathcal{L}(X, \mathcal{Z})$. If S and T are $C^*_{\mathcal{E}}$-upper semicontinuous at $y_0 \in Y$, then so are the functions $S+T$, λT and $T + \ell$, where $\lambda \geqslant 0$ and $\ell \in \mathcal{L}(X, \mathcal{Z})$.*

Proof. It is clear that $T + \ell$ is $C^*_{\mathcal{E}}$-upper semicontinuous at y_0. Let $(E, v) \in \mathcal{E} \times \operatorname{int} C$ be arbitrary. There is a neighborhood U of y_0 such that

$$y \in U \Longrightarrow T(y) \in T(y_0) + \left[E, \frac{v}{\lambda} - \operatorname{int} C\right] \Longrightarrow \lambda T(y) \in \lambda T(y_0) + [E, v - \operatorname{int} C].$$

Therefore, λT is $C^*_{\mathcal{E}}$-upper semicontinuous at y_0.

Finally, we prove that $S+T$ is $C^*_{\mathcal{E}}$-upper semicontinuous at y_0. There is a neighborhood V of y_0 such that

$$y \in V \Longrightarrow S(y) \in S(y_0) + \left[E, \frac{v}{2} - \operatorname{int} C\right] \text{and } T(y) \in T(y_0) + \left[E, \frac{v}{2} - \operatorname{int} C\right].$$

Now, for any $y \in V$,

$$S(y) + T(y) \in S(y_0) + T(y_0) + [E, v - \operatorname{int} C].$$

This completes the proof. □

Let Y be a topological space, and write $X_b^{\mathcal{L}} = \mathcal{L}_b(X, \mathcal{Z})$. Any function f: $Y \longrightarrow X$ induces a function $f^*: Y \longrightarrow \mathcal{L}_b(X_b^{\mathcal{L}}, \mathcal{Z})$ defined by

$$f^*(y) = J(f(y)) \text{ for } y \in Y,$$

where $J(x) = x^*$ for $x \in X$.

In the following, we relate the $C_{\mathcal{L}}$-semicontinuity of f to the $C^*_{\mathcal{L}(b)}$-semicontinuity of f^*.

THEOREM 3.6. *Let f be a function from a topological space Y into X. Then f is $C_{\mathcal{L}}$-upper (respectively, $C_{\mathcal{L}}$-lower) semicontinuous if and only if f^* of f is $C^*_{\mathcal{L}(b)}$-upper (respectively, $C^*_{\mathcal{L}(b)}$-lower) semicontinuous.*

Proof. First, assume that f is $C_{\mathcal{L}}$-upper semicontinuous, and we prove that f^* is $C^*_{\mathcal{L}(b)}$-upper semicontinuous at any fixed point $y_0 \in Y$. Let $\widehat{\mathcal{B}}_X$ denote the family of all bounded subsets of $X^{\mathcal{L}}_b$. Let $(\widehat{E}, v) \in (\widehat{\mathcal{B}}_X, \operatorname{int} C)$ and $\ell \in \widehat{E}$ be arbitrary. Since $\ell \circ f$ is C-upper semicontinuous, there is a neighborhood U of y_0 such that

$$\langle \ell, f(y)\rangle \in \langle \ell, f(y_0)\rangle + v - \operatorname{int} C \text{ for all } y \in U.$$

Since $\langle \ell, f(y)\rangle = \langle f^*(y), \ell\rangle$ for all $y \in Y$, we have

$$\begin{aligned} y \in U &\Longrightarrow \langle f^*(y), \ell\rangle \in \langle f^*(y_0), \ell\rangle + v - \operatorname{int} C \\ &\Longrightarrow \langle f^*(y) - f^*(y_0), \ell\rangle \in v - \operatorname{int} C. \end{aligned}$$

This proves that $f^*(y) \in f^*(y_0) + [\widehat{E}, v - \operatorname{int} C]$ for all $y \in U$. Therefore, f^* is $C^*_{\mathcal{L}(b)}$-upper semicontinuous at y_0.

Conversely, assume that f^* is $C^*_{\mathcal{L}(b)}$-upper semicontinuous. We have to show that $\ell \circ f$ is C-upper semicontinuous for every $\ell \in \mathcal{L}(X, \mathcal{Z})$. Let $(y_0, v) \in Y \times \operatorname{int} C$ be arbitrary. Since $\{\ell\} \in \widehat{\mathcal{B}}_X$, there is a neighborhood U of y_0 such that

$$\begin{aligned} y \in U &\Longrightarrow f^*(y) \in f^*(y_0) + [\{\ell\}, v - \operatorname{int} C] \\ &\Longrightarrow \langle f^*(y) - f^*(y_0), \ell\rangle \in v - \operatorname{int} C \\ &\Longrightarrow \langle \ell, f(y)\rangle \in \langle \ell, f(y_0)\rangle + v - \operatorname{int} C. \end{aligned}$$

The proof is complete. □

THEOREM 3.7. *Let W and Y be topological spaces. For given functions $T\colon W \longrightarrow \mathcal{L}(X, \mathcal{Z})$ and $f\colon Y \longrightarrow X$, let $\Lambda\colon W \times Y \longrightarrow \mathcal{Z}$ be the function defined by*

$$\Lambda(w, y) = \langle T(w), f(y)\rangle \textit{ for } (w, y) \in W \times Y.$$

(i) *If T is $C^*_{\mathcal{L}(b)}$-upper semicontinuous, and if f is $C_{\mathcal{L}}$-upper semicontinuous with $f(Y)$ bounded in X, then Λ is C-upper semicontinuous.*
(ii) *If T is $C^*_{\mathcal{L}(s)}$-upper semicontinuous with $T(W)$ bounded in $\mathcal{L}_b(X, \mathcal{Z})$, and if f is $C_{\mathcal{L}}$-upper semicontinuous, then Λ is C-upper semicontinuous.*

Proof. Let (w_0, y_0) be any fixed point in $W \times Y$, and let $v \in \operatorname{int} C$ be arbitrary. We shall prove that there is a neighborhood Ω of (w_0, y_0) in $W \times Y$ such that

$$\Lambda(w,y) \in \Lambda(w_0,y_0) + v - \operatorname{int} C \quad \text{for all } (w,y) \in \Omega.$$

Note that

$$\Lambda(w,y) - \Lambda(w_0,y_0) = \langle T(w) - T(w_0), f(y)\rangle + \langle T(w_0), f(y) - f(y_0)\rangle.$$

Since the function $y \longmapsto \langle T(w_0), f(y) - f(y_0)\rangle$ is C-upper semicontinuous on Y, there is a neighborhood V of y_0 in Y such that

$$y \in V \Longrightarrow \langle T(w_0), f(y) - f(y_0)\rangle \in \frac{v}{2} - \operatorname{int} C.$$

There is a neighborhood U of w_0 in W such that

$$\begin{aligned} w \in U \Longrightarrow T(w) &\in T(w_0) + \left[f(K), \frac{v}{2} - \operatorname{int} C\right] \\ &\Longrightarrow \langle T(w) - T(w_0), f(y)\rangle \in \frac{v}{2} - \operatorname{int} C \text{ for all } y \in Y. \end{aligned}$$

Now, by setting $\Omega = U \times V$, we have

$$\Lambda(w,y) - \Lambda(w_0,y_0) \in v - \operatorname{int} C \quad \text{for all } (w,y) \in \Omega.$$

Next, we prove (ii). Let (w_0, y_0) and v be given above. Note that

$$\begin{aligned} \Lambda(w,y) - \Lambda(w_0,y_0) &= \langle T(w) - T(w_0), f(y_0)\rangle + \langle T(w), f(y) - f(y_0)\rangle \\ &= \langle T(w) - T(w_0), f(y_0)\rangle + \langle f^*(y) - f^*(y_0), T(w)\rangle. \end{aligned}$$

There is a neighborhood U of w_0 in W such that

$$\begin{aligned} w \in U \Longrightarrow T(w) - T(w_0) &\in \left[\{f(y_0)\}, \frac{v}{2} - \operatorname{int} C\right] \\ &\Longrightarrow \langle T(w) - T(w_0), f(y_0)\rangle \in \frac{v}{2} - \operatorname{int} C. \end{aligned}$$

By Theorem 3.6, there is a neighborhood V of y_0 in Y such that

$$\begin{aligned} y \in V \Longrightarrow f^*(y) - f^*(y_0) &\in \left[T(W), \frac{\nu}{2} - \operatorname{int} C\right] \\ &\Longrightarrow \langle f^*(y) - f^*(y_0), T(w)\rangle \in \frac{\nu}{2} - \operatorname{int} C \quad \text{for all } w \in W. \end{aligned}$$

Now,

$$\Lambda(w,y) - \Lambda(w_0,y_0) \in \nu - \operatorname{int} C \quad \text{for all } (w,y) \in U \times V.$$

Therefore, Λ is C-upper semicontinuous at (w_0, y_0). □

REMARK 3.3. By use of notation given in Theorem 3.7, assume that $W = Y$. We remark that if Λ is C-upper semicontinuous, then the function $g: Y \longrightarrow \mathcal{Z}$ defined by

$$g(y) = \langle T(y), f(y) \rangle$$

is also C-upper semicontinuous.

REMARK 3.4. Let T, f and Λ be given in Theorem 3.7. From Proposition 2.3 we conclude that Λ is continuous if either one of the following holds.

(a) $T: W \longrightarrow \mathcal{L}_b(X, \mathcal{Z})$ and $f: Y \longrightarrow X_{\mathcal{L}}$ (or $f: Y \longrightarrow X$ are continuous, and $f(Y)$ is bounded in X.
(b) $T: W \longrightarrow \mathcal{L}_s(X, \mathcal{Z})$ and $f: Y \longrightarrow X$ are continuous, and $T(W)$ is bounded in $\mathcal{L}_b(X, \mathcal{Z})$.

COROLLARY 3.1. *Let W and Y be topological spaces, let $T: W \longrightarrow \mathcal{L}(X, \mathcal{Z})$ and $f: Y \longrightarrow X$ be functions, and let Λ be the function defined in Theorem* 3.7.

(i) *Assume that T is $C^*_{\mathcal{L}(b)}$- upper semicontinuous, and f is continuous. If Y is compact, then Λ is C-upper semicontinuous.*
(ii) *Assume that $T: W \longrightarrow \mathcal{L}_b(X, \mathcal{Z})$ is continuous, and $f: Y \longrightarrow X$ is $C_{\mathcal{L}}$-upper semicontinuous. If W is compact, then Λ is C-upper semicontinuous.*

COROLLARY 3.2. *Assume that $\mathcal{Z}$ is a Banach space and X is a normed space, and let W and Y be topological spaces with Y compact. If $T: W \longrightarrow \mathcal{L}(X, \mathcal{Z})$ is $C^*_{\mathcal{L}(b)}$-upper semicontinuous, and if $f: Y \longrightarrow X_{\mathcal{L}}$ is continuous, then the function Λ defined in Theorem* 3.7 *is C-upper semicontinuous.*

Proof. Since $f(Y)$ is $\mathcal{L}$-compact and $\mathcal{L}$-bounded in X, by Proposition 2.2, $f(Y)$ is bounded in X. Now, the corollary follows from Theorem 3.7. □

COROLLARY 3.3. *Assume that either X and $\mathcal{Z}$ are Hausdorff locally convex with X barrelled, or X is a Baire space. Let $T: W \longrightarrow \mathcal{L}(X, \mathcal{Z})$ and $f: Y \longrightarrow X$ be functions. Then the function Λ defined in Theorem* 3.7 *is C-upper semicontinuous if either*

(i) *T is $C^*_{\mathcal{L}(s)}$-upper semicontinuous with $T(W)$ bounded in $\mathcal{L}_s(X, \mathcal{Z})$, and $f: Y \longrightarrow X$ is $C_{\mathcal{L}}$-upper semicontinuous, or*
(ii) *W is compact, $T: W \longrightarrow \mathcal{L}_s(X, \mathcal{Z})$ is continuous, and $f: Y \longrightarrow X$ is $C_{\mathcal{L}}$-upper semicontinuous.*

Proof. It follows from Corollary 4.1 and Theorem 4.2 (p. 83) of Schaefer (1999) that $T(W)$ is equicontinuous on X and is bounded in $\mathcal{L}_b(X, \mathcal{Z})$. The proof is complete. □

REMARK 3.5. Let X and $\mathcal{Z}$ be given in Corollary 3.3. If $T: W \longrightarrow \mathcal{L}_s(X, \mathcal{Z})$ is continuous with $T(W)$ bounded in $\mathcal{L}_s(X, \mathcal{Z})$, and if $f: Y \longrightarrow X$ is continuous, then the function Λ defined in Theorem 3.7 is continuous. See Remark 2.1.

4. Mixed Vector Variational-like Inequalities

This section is devoted to deriving existence results for the mixed vector variational-like inequality given in (1). The results will be established by using an existence result for equilibrium problems obtained in Chadli et al. (2002), stated below as Lemma 4.1.

To state the results, we need some basic definitions. Let K be a nonempty convex subset of X, and let $g: K \longrightarrow \mathcal{Z}$ and $h: K \times K \longrightarrow \mathcal{Z}$ be functions.

(i) g is called C-convex (Chen and Li, 1996) if

$$g((1-t)x_0 + tx_1) \in (1-t)g(x_0) + g(x_1) - C$$

whenever $x_0, x_1 \in K$ and $0 \leqslant t \leqslant 1$.

(ii) For any fixed $y \in K$, we shall use h_y for the function on K defined by

$$h_y(x) = h(x, y) \quad \text{for } x \in K.$$

(iii) h is called *vector* 0-*diagonally convex* (Chadli et al., 2002) if for any finite set $\{y_1, \dots, y_m\} \subset K$,

$$x = \sum_{j=1}^{m} t_j y_j \text{ with all } t_j \geqslant 0 \text{ and} \sum_{j=1}^{m} t_j = 1 \Longrightarrow \sum_{j=1}^{m} t_j h(x, y_j) \in (-\operatorname{int} C)^c.$$

When $\mathcal{Z} = \mathbb{R}$, the notion of vector 0-diagonally convexity reduces to the notion of 0-diagonally convexity introduced in Zhow and Chen (1998).

(iv) h is said to satisfy the (L)-condition if for any $x, y \in K$ and any net $\{x_\alpha\}$ in K converging to x, the following implication holds:

$$h(x_\alpha, (1-t)x + ty) \in (-\operatorname{int} C)^c \text{ for all } \alpha \text{ and } 0 \leqslant t \leqslant 1 \Longrightarrow h(x, y) \in (-\operatorname{int} C)^c.$$

REMARK 4.1. Let K be a convex subset of X, and let $h: K \times K \longrightarrow \mathcal{Z}$ be a function.

(a) h satisfies the (L)-condition if for every $y \in K, h_y$ is C-upper semicontinuous. See Corollary 2.6 and Theorem 2.7 of Chadli et al. (2002).
(b) If h is vector 0-diagonally convex, and if $g: K \longrightarrow \mathcal{Z}$ is C-convex, then the function

$$(x, y) \longmapsto h(x, y) + g(y) - g(x)$$

is vector 0-diagonally convex (cf. Ansari and Yao, 2001).

The following is stated as Lemma 3.5 in Chadli et al. (2002):

LEMMA 4.1. *Let K be a nonempty convex subset of a Hausdorff topological vector space X, and let $f: K \times K \longrightarrow \mathcal{Z}$ be a bifunction. Assume that the following conditions hold.*

(i) *f is vector 0-diagonally convex and satisfies the (L)-condition.*
(ii) *For every $y \in K$, f_y is C-upper semicontinuous on $co(E)$ for every $E \in \mathcal{F}(K)$.*
(iii) *(Coercivity) There exist a nonempty compact set $K_0 \subset K$ and a nonempty convex compact set $K_1 \subset K$ such that*

$$x \in K \backslash K_0 \Longrightarrow f(x, y_x) \in (-\operatorname{int} C) \text{ for some } y_x \in K_1.$$

Then there is an $\hat{x} \in K$ such that $f(\hat{x}, y) \in (-\operatorname{int} C)^c$ for all $y \in K$.

THEOREM 4.1. *Let X be Hausdorff, let K be a nonempty convex subset of X, and let*

$$T: K \longrightarrow \mathcal{L}(X, \mathcal{Z}), \quad \varphi: K \times K \longrightarrow X \text{ and } f: K \times K \longrightarrow \mathcal{Z}$$

be functions. Assume that the following conditions are satisfied.

(i) *$f(x, x) = 0$ for all $x \in K$.*
(ii) *For every $x \in K$, the function $y \longmapsto f(x, y)$ is C-convex.*
(iii) *The function $\Lambda: K \times K \longrightarrow \mathcal{Z}$ defined by*

$$\Lambda(x, y) = \langle Tx, \varphi(x, y) \rangle \text{ for } (x, y) \in K \times K,$$

is vector 0-diagonally convex.
(iv) *The function $h = \Lambda + f$ satisfies the (L)-condition.*
(v) *(Coercivity) There exist a nonempty compact set $K_0 \subset K$ and a nonempty convex compact set $K_1 \subset K$ such that*

$$x \in K \backslash K_0 \Longrightarrow \langle Tx, \varphi(x, y_x) \rangle + f(x, y_x) \in (-\operatorname{int} C) \text{ for some } y_x \in K_1.$$

(vi) *For every* $y \in K$ *and every* $E \in \mathcal{F}(K)$,

(a) φ_y *is* $C_{\mathcal{L}}$*-upper semicontinuous on* $co(E)$ *with* $\varphi_y(co(E))$ *bounded in* X;
(b) T *is* $C^*_{\mathcal{L}(b)}$*-upper semicontinuous on* $co(E)$;
(c) f_y *is* C*-upper semicontinuous on* $co(E)$.

Then the MVVLI (T, φ, f) *has a solution.*

Proof. From Theorem 3.7 and Remark 3.4, we conclude that for every $y \in K$ and every $E \in \mathcal{F}(K)$, Λ_y is C-upper semicontinuous on $co(E)$, and so is the function h_y. By Lemma 4.1, it remains to show that h is vector 0-diagonally convex.

Let $\{y_1, \dots, y_m\}$ be any finite subset of K, and let $x = \sum_{j=1}^m t_j y_j$ with all $t_j \geqslant 0$ and $\sum_{j=1}^m t_j = 1$. It follows from the condition (i) and (ii) that

$$0 = f(x, x) = f\left(x, \sum_{j=1}^m t_j y_j\right) \in \sum_{j=1}^m t_j f(x, y_j) - C \text{ and } \sum_{j=1}^m t_j f(x, y_j) = c$$

for some $c \in C$. If $\sum_{j=1}^m t_j h(x, y_j) \in (-\text{int}\, C)$, then

$$\sum_{j=1}^m t_j \Lambda(x, y_j) = \sum_{j=1}^m t_j \{h(x, y_j) - f(x, y_j)\} = -c + \sum_{j=1}^m t_j h(x, y_j) \in (-\text{int}\, C).$$

This contradicts to the condition (iii). The proof is complete. □

REMARK 4.2. Use notation given in Theorem 4.1.

(a) If for every $x \in K, \varphi(x, x) = 0$ and the function $y \longmapsto \Lambda(x, y)$ is C-convex, then $\Lambda(x, y)$ is vector 0-diagonally convex.
(b) If K is compact, then in Theorem 4.1, the condition (v) can be omitted.
(c) If T is $C^*_{\mathcal{L}(b)}$-upper semicontinuous on K, and if for every $y \in K, f_y$ is C-upper semicontinuous on K and φ_y is $C_{\mathcal{L}}$-upper semicontinous on K with $\varphi_y(K)$ bounded in X, then in Theorem 4.1, the condition (iv) can be omitted. See Remark 4.1 (a).
(d) If for every $y \in K, \varphi_y$ is continuous, then in Theorem 4.1, the boundedness assumption on $\varphi_y(\text{co}(E))$ can be omitted.

By using Theorem 3.7(ii), the same reasoning as above proves:

THEOREM 4.2. *Let X be Hausdorff, let K be a nonempty convex subset of X, and let*

$$T: K \longrightarrow \mathcal{L}(X, \mathcal{Z}), \quad \varphi: K \times K \longrightarrow X \text{ and } f: K \times K \longrightarrow \mathcal{Z}$$

be functions satisfying the conditions (i)–(v) *of Theorem* 4.1. *If for every $y \in K$ and every $E \in \mathcal{F}(K)$,*

(a) *φ_y is $C_{\mathcal{L}}$-upper semicontinuous on $co(E)$;*
(b) *T is $C^*_{\mathcal{L}(s)}$-upper semicontinuous on $co(E)$ with $T(co(E))$ bounded in $\mathcal{L}_b(X, \mathcal{Z})$;*
(c) *f_y is C-upper semicontinuous on $co(E)$,*

then the MVVLI (T, φ, f) *has a solution.*

From Corollary 3.3, we obtain:

THEOREM 4.3. *Assume that either X and $\mathcal{Z}$ are either Hausdorff locally convex with X barrelled, or X is a Baire space. Let K be a nonempty convex subset of X, and let*

$$T: K \longrightarrow \mathcal{L}(X, \mathcal{Z}), \quad \varphi: K \times K \longrightarrow X \text{ and } f: K \times K \longrightarrow \mathcal{Z}$$

be functions satisfying the conditions (i)–(v) *of Theorem* 4.1. *If for every $y \in K$ and every $E \in \mathcal{F}(K)$,*

(a) *φ_y is $C_{\mathcal{L}}$-upper semicontinuous on $co(E)$;*
(b) *T is $C^*_{\mathcal{L}(s)}$-upper semicontinuous on $co(E)$ with $T(co(E))$ bounded in $\mathcal{L}_s(X, \mathcal{Z})$;*
(c) *f_y is C-upper semicontinuous on $co(E)$,*

then the MVVLI (T, φ, f) *has a solution.*

References

1. Ansari, Q.H. and Yao, J.C. (2001), Iterative-schemes for solving mixed variational-like inequalities, *Journal of Optimization Theory and Applications*, 108, 521–529.
2. Ansari, Q.H. and Yao, J.C. (2000), Nonlinear variational inequalities for pseudomonotone operators with applications, *Advances in Nonlinear Variational Inequalities*, 3, 61–69.
3. Ansari, Q.H. and Yao, J.C. (1998), Prevariational inequalities in banach spaces. In Cacetta, L. et al. (eds.), *Optimization Techniques and Applications*, Curtin University of Technology, Perth, Australia, Vol. 2, pp. 1165–1172.
4. Chadli, O., Chiang, Y. and Huang, S. (2002), Topological pseudomonotonicity and vector equilibrium problems, *Journal of Mathematical Analysis and Applications*, 270, 435–450.
5. Chen, G.Y. (1989), Vector variational inequality and its applications for multiobjective optimization, *Chinese Science Bulletin*, 34, 969–972.

6. Chen, G.Y. and Yang, X.Q. (1990), Vector complementarity problem and its equivalence with weak minimal element in ordered spaces, *Journal of Mathematical Analysis and Applications*, 153, 136–158.
7. Chen, G.Y. and Cheng, G.M. (1987), Vector variational inequalities and vector optimization, *Lecture Notes in Economics and Mathematical Systems*, Springer-Verlag, New York/Berlin, 285, 408–416.
8. Chen, G.Y. and Li, S.L. (1996), Existence of solutions for a generalized quasi-vector variational inequalities, *Journal of Optimization Theory and Applications*, 90, 321–334.